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A fractional-order fall armyworm-maize biomass model with naturally beneficial insects and optimal farming awareness

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ABSTRACT

Maize remains an important food crop in Africa. However, the production of this crop, and consequently the livelihood of the growers are threatened by the invasion and widespread infestation of the fall armyworm which causes substantial maize yield losses. In this paper, a fractional-order fall armyworm-maize biomass model with naturally beneficial insects and optimal farming awareness has been formulated. Comprehensive analysis of the model has shown that it contains five equilibrium points which are all locally and globally asymptotically stable if the conditions outlined in Lemma 2.1 and 2.2 are met. We also carried out numerical simulations to support the analytical results and to illustrate different dynamical regimes that can be observed in the model. We have found that time-dependent farming awareness can significantly reduce fall armyworm population if the cost of implementation is relatively low.

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1. Introduction

In the last half century, invasive species have caused unprecedented challenges to agricultural systems globally. In sub-Saharan Africa (SSA), agriculture is considered the primary source of livelihoods for most households [1,2]. However, its contribution to food security and poverty reduction is hampered by several, often interacting, biotic and abiotic factors. For instance, the recent invasion of fall armyworm (FAW-*Spodoptera frugiperda* JE Smith) in SSA has become a major threat to food security in the region [2,3]. The first outbreak of FAW in Africa occurred in West Africa in 2016, and to date the pest has spread to 44 countries in the continent [2]. The FAW can cause damage to more than 80 crop species, including economically important crops such as maize, rice, sorghum, wheat, sugarcane and cotton just to mention a few.

Current estimates from 12 African countries suggest an annual loss of (4.1–17.6) million tons of maize due to FAW infestations [2]. In particular, farm-level estimates from Ghana and Zambia suggest yield losses of (22–67) per cent [3], 47% in Kenya [4] and 9.4% in Zimbabwe [5] due to FAW infestations. In maize, FAW attacks all cropping stages from

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seedling emergence through to ear development. They defoliate and destroy young plants whereby, whorl damage can result in yield losses, and ear feeding can result in the reduction of grain quality and yields [6]. The Management of the FAW involves the integration of several approaches, including the use of insecticides, host plant resistance, and biological control. However, all these approaches depend on several characteristics of the involved agro-ecosystems [7]. In South America where the pest has been a challenge for quite sometime, the common management strategy has been the use of insecticide sprays and genetically modified crops like Bt maize [6].

Due to financial challenges associated with most of African governments, alongside the cost associated with massive spraying programs of chemical insecticides and the use of genetically modified crops like Bt maize, the effective management of this pest in the continent remains a challenge [6]. In addition, excessive use of chemical insecticides is associated with negative environmental effects and can lead to the development of pest resistance [8]. At the backdrop of this, integrated pest management (IPM) has been gaining more attention among researchers and its application is also increasing the crop yields [9,10]. This approach seeks to minimize the reliance on pesticides use by emphasizing the application of biological control agents.

Mass media can affect the spread and attack poised by FAW during an outbreak. Furthermore, awareness campaigns, particularly through various media outlets such as radio, newspapers, TV and so on, do not only make farmers aware of FAW outbreaks but also improve trust on IPM a nation will be advocating for. In recent times, attention to health news has been observed to play an integral role in disease management [11]. There is no doubt that correct and relevant knowledge about maize crop and its pests is essential to farmers [10].

The main goal of this paper is to develop a mathematical model to assess the effects of media campaigns during a FAW outbreak. Mathematical models of plant-pest interactions have provided insights into effective methods for effective pest management as well as way of increasing plant productivity (e.g. [10,12–22]). In some of the studies (e.g. [10,12–14,16,23]), mathematical models were used to investigate the effects of biological control on the dynamics of plant pest interactions, while in other studies (e.g. [17–22]), pest management models based solely on chemical controls were proposed and analyzed. For instance, Liu and Teng [18] utilized a mathematical model to assess the impact of spraying pesticides at a fixed time on the pest reproductive cycles. Among several outcomes, their study showed that there exists an optimal time for pest control if the pesticides were to be applied just before each birth pulse of the cycle. Wei [24] proposed pest control models that incorporated birth pulse and were based on the assumption that pesticides killed adult pests or larvae or both. Making use of numerical simulation, the study demonstrated that with the different elimination rates for larvae and adults, the corresponding optimal times for pesticide applications were also different.

These studies and several others (e.g. [10,12–14,16–22,25]) have certainly produced many useful results and improved the existing knowledge on plant-pest interaction dynamics. Despite of all these efforts, mathematical models for FAW management during an outbreak are very few and of the few that exists there are some limitations; (i) majority of these few were general and not pest-specific, which implies that their results were also general. Practically, pests are not general, rather, they follow different biological development cycles, hence more informative plant-pest interaction models need to be pest-specific and closely follow the life cycle of the pest involved (ii) the presented models utilized integer-order differential equations (IDEs) which according to Caputo [26], do not replicate real-world problems nor capture memory effects as compared to fractional calculus.

Furthermore, unlike IDEs, models based on fractional calculus have been found to be more accurate with regard to describing rules and development processes of several phenomena in natural science [27] and this has been attributed to the fact that fractional order models possess memory effects and hereditary properties. Hence, there has been growing interest among researchers to use fractional calculus in modeling real-world problems, and some remarkable achievements have been made [27]. Cognizant of this, a fractional order pest-plant based model has been proposed in the present study with the aim to study the effects of educational campaigns and FAW larval predation on persistence and extinction of the pest in a maize field. The model incorporates the maize biomass and two essential development stages of the FAW, that is, the larval and the moth (adult). In addition, since FAW larvae are prey to several parasitoids, predators and pathogens like birds, rodents, beetles, earwigs [28], the proposed model incorporates the predator population.

The rest of the paper is organized as follows: In Section 2, a fractional-order FAW model is proposed and analyzed. In particular, the model's steady states have been computed and their stability has been investigate as well. In Section 3, we perform an optimal control study to determine the effects of farming on minimizing the effects of FAW on maize biomass, through both mathematical analysis and numerical simulation. Finally, we conclude the paper with some discussion in Section 4.

2. Model formulation and analytical results

2.1. Model formulation

We developed a mathematical model for FAW outbreak in a maize field focusing on investigating the effects of farming awareness and biological control (FAW predators). The proposed model is based on fractional calculus of Caputo type [26]. The FAW population is subdivided into two classes; the larvae $L(t)$ and the adult which also known as the moth $A(t)$. The FAW predator population is modeled by $Z(t)$. Meanwhile, the dynamics of maize biomass are represented by $M(t)$. The proposed model is governed by the following assumptions:

(i) We assume logistic growth for the density of maize biomass, with net growth rate r and carrying capacity K_M. Let β be the consumption rate by FAW larvae and e be the efficiency of biomass conversion. Awareness is assumed to reduce the attack rate of the maize crop by FAW larvae by a factor $1-u$, with $0 \leq u \leq 1$. Thus $u=0$ implies that awareness has no impact on reducing the attack rate of the maize plant by FAW whereas $u=1$ implies that farming awareness is 100% efficient in protecting the maize crop from FAW attack during an outbreak.

(ii) The dynamics of the FAW larvae are assumed to follow a logistic growth model, with net growth rate b_L and the carrying capacity K_L. The larvae are assumed to progress to the adult stage after approximately $1/\alpha_L$ days. The FAW larvae and adults suffer natural mortality at rates μ_L and μ_A, respectively. Apart from natural mortality, both populations diminish due to mortality attributed to the mitigation strategies carried out by farmers as a result of awareness, at the rate ud, where d is the mortality rate of the FAW larvae and adult. Note that if awareness does not have an impact ($u=0$) on FAW populations, then these populations suffer natural mortality only.

(iii) Even though biological control may not replace conventional insecticides, a number of parasitoids, predators and pathogens like birds, rodents, beetles and earwigs readily attack the larvae [28]. To account for the effect of larval predation, let σ be the attack rate of the larvae by predators and ρ be the efficiency of conversion. The average life span of predators is assumed to be $1/\eta$ days.

Based on the above assumptions, the proposed model is summarized by the following system of nonlinear ordinary differential equations (Fig. 1 shows the transition diagram):

$$\left.\begin{array}{rcl} {}_a^cD_t^qM(t) &=& r^qM\left(1-\dfrac{M}{K_M^q}\right)-\beta^q(1-u)LM, \\ {}_a^cD_t^qL(t) &=& b_L^qA\left(1-\dfrac{L}{K_L^q}\right)+e\beta^q(1-u)LM-\sigma^qZL-(\mu_L+\alpha_L+ud)L, \\ {}_a^cD_t^qA(t) &=& \alpha_L^qL-(\mu_A^q+ud)A, \\ {}_a^cD_t^qZ(t) &=& \rho\sigma^qLZ-\eta^qZ. \end{array}\right\} \tag{1}$$

with initial conditions as:

$$M(0)\geq 0, \qquad L(0)\geq 0, \qquad A(0)\geq 0, \qquad Z(0)\geq 0. \tag{2}$$

Here, ${}_a^cD_t^q$ represents the Caputo fractional derivative of order q ($0<q<1$). The Caputo fractional derivative of order q is defined [29]:

$${}_a^cD_t^qf(t)=\frac{1}{\Gamma(n-q)}\int_0^t\frac{f^n(\xi)}{(t-\xi)^{q+1-n}}d\xi, \quad n-1<q<n\in\mathbb{N},$$

where Γ represents the gamma function and the Riemann Liouville fractional integral of arbitrary real order $q>0$ of a function $f(t)$ is defined by the following integral:

$$J^qf(t)=\frac{1}{\Gamma(q)}\int_0^t(t-\xi)^{q-1}f(\xi)d\xi,$$

$J^0f(t)=f(t)$.

Remark 2.1. Note that, in order to avoid flaws regarding the time dimension, we introduced q in the model parameters (right-hand side) of system (2) so that the dimensions of these parameters became $(time)^{-q}$ which is in agreement with the left-hand side of the model.

2.2. Positivity and boundedness of solutions

In this section, we study the positivity and boundedness of solutions of the proposed fractional order model (2) to establish if it is mathematically and biological poised. It follows from (2) that:

Theorem 2.1. *Model (2) is positively invariant and bounded in $\mathbb{R}_+^4$.*

Proof. This begin by demonstrating that $\mathbb{R}_+^4=\{(M,L,A,Z)\in\mathbb{R}_+^4: M(0)\geq 0, L(0)\geq 0, A(0)\geq 0, Z(0)\geq 0\}$ is positively invariant. For that, we demonstrated that on each hyper-plane bounding the non-negative orthant, the vector field points to $\mathbb{R}_+^4$. Therefore, for $M(0)\geq 0$, $L(0)\geq 0$, $A(0)\geq 0$, $Z(0)\geq 0$, we have

$$\left.\begin{array}{rcl} {}_a^cD_t^qM(t)\,|_{M=0} &=& 0, \\ {}_a^cD_t^qL(t)\,|_{L=0} &=& b_L^qA\geq 0, \\ {}_a^cD_t^qA(t)\,|_{A=0} &=& \alpha_L^qL\geq 0, \\ {}_a^cD_t^qZ(t)\,|_{Z=0} &=& 0. \end{array}\right\} \tag{3}$$

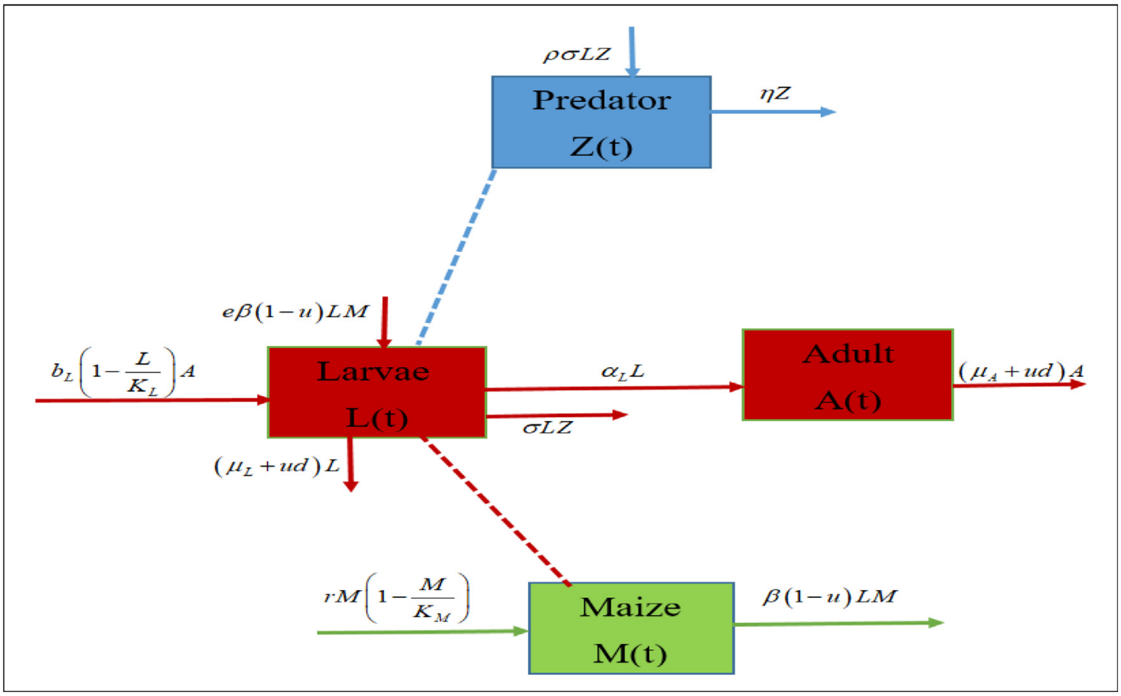

Fig. 1. Model flow diagram illustrating the dynamics of the FAW in a maize field. The fAW life cycle is divided into two classes; the larvae $L(t)$ and adult $A(t)$ population. The FAW predator and maize biomass population are represented by compartment $Z(t)$ and $M(t)$ respectively. Continuous lines indicate either inflow or outflow transition between compartments. Red and blue discontinuous arrows connecting compartment $L(t)$ with compartments $Z(t)$ and $M(t)$ show the interaction that occurs between the predators $Z(t)$ and FAW larvae $L(t)$ as well as with maize biomass $M(t)$. Note that the predator has an effect on larvae which in turn have an effect on maize biomass. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Based on the results in (3), it follows that model (2) is positively invariant in $\mathbb{R}^4_+$. Further, from the first equation of model (2), we demonstrate that $M(t) \leq K_M$, $\forall t \geq 0$. If there exists t_0 such that $M(t_0) > K_M$, then due to the continuity of $M(t)$:

$$\exists B_\epsilon(t_0) : \forall t \in B_\epsilon(t_0) : M(t) > K_M, \tag{4}$$

so:

$$rM\left(1 - \frac{M}{K_M}\right) < 0. \tag{5}$$

Thus ${}^{c}_{t_0}D^q M(t) < 0$. From the continuity of $M(t)$ and $\frac{dM}{dt} = \lim_{q\to 1^-} {}^{c}_{t_0}D^q M(t) < 0$, hence we conclude that $M(t)$ is a decreasing function for all $t \geq 0$ and it follows that $0 \leq M(t) \leq M(0) \leq K_M$, $\forall t \geq 0$, and this is a contradiction to (4). Thus $M(t) \leq K_M$, for all $t \geq 0$. Using a similar approach it can easily be verified that $0 \leq L(t) \leq K_L$. Now, from the third equation of system (2) we have:

$$\begin{aligned} {}^{c}_{a}D^q_t A(t) &= \alpha_L^q L - (\mu_A^q + ud^q)A \\ &\leq \alpha^q K_L^q - (\mu_A^q + ud^q)A. \end{aligned} \tag{6}$$

Applying the Laplace transform one gets:

$$s^q \mathcal{L}[A(t)] - s^{q-1}A(0) \leq \frac{\alpha_L^q K_L^q}{s} - (\mu_a^q + ud^q)\mathcal{L}[A(t)]. \tag{7}$$

After combining like terms one gets:

$$\begin{aligned} \mathcal{L}[A(t)] &\leq \alpha_L^q K_L^q \frac{s^{-1}}{s^q + (\mu_a^q + ud^q)} + A(0)\frac{s^{q-1}}{s^q + (\mu_A^q + ud)} \\ &= \alpha_L^q K_L^q \frac{s^{q-(1+q)}}{s^q + (\mu_a^q + ud^q)} + A(0)\frac{s^{q-1}}{s^q + (\mu_a^q + ud^q)}. \end{aligned} \tag{8}$$

Applying the inverse Laplace transform leads to:

$$\begin{aligned}
A(t) &\leq \mathcal{L}^{-1}\left\{\alpha_L^q K_L^q \frac{s^{q-(1+q)}}{s^q+(\mu_a^q+ud^q)}\right\}+A(0)\mathcal{L}^{-1}\left\{\frac{s^{q-1}}{s^q+(\mu_a^q+ud^q)}\right\} \\
&\leq \alpha_L^q K_L^q t^q E_{q,q+1}(-(\mu_a^q+ud^q)t^q)+A(0)E_{q,1}(-(\mu_a^q+ud^q)t^q) \\
&\leq \frac{\alpha_L^q K_L^q}{(\mu_A^q+ud)}(\mu_A^q+ud)t^q E_{q,q+1}(-(\mu_a^q+ud^q)t^q)+A(0)E_{q,1}(-(\mu_a^q+ud^q)t^q) \\
&\leq \max\left\{\frac{\alpha^q K_L^q}{(\mu_a^q+ud^q)}, A(0)\right\}((\mu_a^q+ud^q)t^q E_{q,q+1}(-(\mu_a^q+ud^q)t^q)+E_{q,1}(-(\mu_a^q+ud^q)t^q)) \\
&= \frac{C}{\Gamma(1)} = C_A,
\end{aligned} \tag{9}$$

where E_q is the Mittag-Leffler function and $C_A = \max\left\{\frac{\alpha_L^q K_L^q}{(\mu_a^q+ud^q)}, P(0)\right\}$. Thus, $A(t)$ is bounded from above. From the last equation of system (2) we have:

$$\begin{aligned}
{}_a^cD_t^q Z(t) &= \rho\sigma^q LZ - \eta^q Z \\
&\leq -(\eta^q - \rho\sigma^q K_L^q)Z.
\end{aligned} \tag{10}$$

Applying the Laplace transform in the previous inequality, we get:

$$s^q\mathcal{L}[Z(t)] - s^{q-1}Z(0) \leq -(\eta^q-\rho\sigma^q K_L^q)\mathcal{L}[Z(t)], \tag{11}$$

which can be written as:

$$\mathcal{L}[Z(t)] \leq Z(0)\frac{s^{q-1}}{s^q+(\eta^q-\rho\sigma^q K_L^q)}. \tag{12}$$

Applying the inverse Laplace transforms leads to

$$Z(t) \leq Z(0)E_q[-(\eta^q-\rho\sigma^q K_L^q)t^q]. \tag{13}$$

Hence, we conclude that $Z(t)$ is bounded. □

2.3. Equilibrium points and their existence

The fractional-order model (2) has the following six equilibrium points:

(a) The trivial equilibrium point $\mathcal{E}_0 : (M_0, L_0, A_0, Z_0) = (0, 0, 0, 0)$ always exists.

(b) The pest-extinction equilibrium point $\mathcal{E}_1 : (M_1, L_1, A_1, Z_1) = (K_M, 0, 0, 0)$ always exists.

(c) The plant-extinction equilibrium point $\mathcal{E}_2 : (M_2, L_2, A_2, Z_2)$ where:

$$\left.\begin{aligned}
M_2 &= 0, \qquad L_2 = \frac{\eta^q}{\rho}, \qquad A_2 = \frac{\eta^q\alpha^q}{(\mu_A^q+ud^q)\rho}, \\
Z_2 &= \frac{b^q\eta^q+\rho K_L^q(\mu_L^q+\alpha^q+ud^q)}{\rho\sigma^q K_L^q}\left(\frac{b^q\rho K_L^q}{b^q\eta^q+\rho K_L^q(\mu_L^q+ud^q+\alpha^q)}-1\right).
\end{aligned}\right\} \tag{14}$$

Thus, the equilibrium point $\mathcal{E}_2$ makes biological sense if $\frac{b^q\rho K_L^q}{b^q\eta^q+\rho K_L^q(\mu_L^q+ud^q+\alpha_L^q)} > 1$.

(d) The plant and predator-extinction equilibrium point $\mathcal{E}_3 : (M_3, L_3, A_3, Z_3)$ where:

$$\left.\begin{aligned}
M_3 &= 0, \qquad L_3 = \frac{\eta^q K_L^q}{b^q}\left(\frac{b^q}{(\mu_L^q+\alpha_L^q+ud^q)}-1\right), \\
A_3 &= \frac{\alpha_L^q K_K^q(\mu_L^q+\alpha_L^q+ud^q)}{b^q(mu_A^q+ud^q)}\left(\frac{b^q}{(\mu_L^q+\alpha_L^q+ud^q)}-1\right), \\
Z_3 &= 0.
\end{aligned}\right\} \tag{15}$$

Therefore the equilibrium point $\mathcal{E}_3$ exists and is biologically meaningful if $b^q > (\mu_L^q+\alpha_L^q+ud^q)$.

(e) The predator-extinction equilibrium point is $\mathcal{E}_4 : (M_4, L_4, A_4, Z_4)$ where:

$$\left.\begin{aligned}
M_4 &= \frac{K_M^q\left[r^q b^q + \beta^q K_L^q(\mu_L^q + \alpha_L^q + ud^q)\left(1 - \dfrac{b^q}{(\mu_L^q + \alpha_L^q + ud^q)}\right)\right]}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q},\\
L_4 &= \frac{r^q K_L^q}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right),\\
A_4 &= \frac{r^q \alpha^q k_L^q}{(\mu_A^q + ud^q)(\mu_L^q + \alpha_L + ud^q)\tilde{n}}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1\right),\\
Z_4 &= 0.
\end{aligned}\right\} \tag{16}$$

where $\tilde{n} = (r^q b^q + e(\beta^q(1-u))^2 K_L^q K_M^q$. It follows that the equilibrium point $\mathcal{E}_4$ exists and is biologically feasible if $b^q + e\beta^q(1-u)K_M^q > (\mu_L^q + \alpha_L^q + ud^q)$ with $b^q < (\mu_L^q + \alpha_L^q + ud^q)$.

(f) The coexistence equilibrium point $\mathcal{E}_5 : (M_5, L_5, A_5, Z_5)$ where:

$$\left.\begin{aligned}
M_5 &= \frac{K_M^q \beta^q \eta^q}{r^q \rho}\left(\frac{r^q \rho}{\beta^q \eta^q} - 1\right) \qquad L_5 = \frac{\eta^q}{\eta^q}, \qquad A_5 = \frac{\eta^q \alpha_L^q}{(\mu_A^q + ud^q)\rho},\\
Z_5 &= \frac{K_M^q e(\beta^q(1-u))^2 \eta^q}{r^q \rho \sigma^q}\left(\frac{r^q \rho^q}{\beta^q \eta^q} - 1\right)\\
&\quad + \frac{\sigma^q(K_L^q r^q \rho(\mu_L^q + \alpha_L^q + ud^q)) + b^q}{r^q \rho}\left(\frac{b^q K_L^q r^q \rho}{b^q r^q \eta^q(\mu_L^q + \alpha_L^q + ud^q)K_L^q r^q \rho} - 1\right).
\end{aligned}\right\} \tag{17}$$

Therefore, the equilibrium point $\mathcal{E}_5$ exists and is biologically meaningful if $b^q K_L^q r^q \rho > b^q r^q \eta^q(\mu_L^q + \alpha_L^q + ud^q)K_L^q r^q \rho$ with $r^q \rho > \beta^q \eta^q$.

2.4. Local stability analysis of the equilibrium points

The local stability analysis of for the fractional order model (2) around the above equilibrium points is obtained by computing the Jacobian matrix corresponding to equilibrium points. The Jacobian matrix of system (2) is as follows:

$$J(M, L, A, Z) = \begin{bmatrix} r^q - \beta^q L - \dfrac{2r^q M}{K_M^q} & -\beta^q M & 0 & 0 \\ e\beta^q(1-u)L & n & 0 & -\sigma^q L \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & \rho Z & 0 & -\eta^q + \rho L \end{bmatrix}. \tag{18}$$

with $n = b^q + e\beta^q(1-u)M - \sigma^q Z - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q L}{K_L^q}$. The local stability of the equilibrium points of model (2) is now investigated making use of the Jacobian matrix (18) and Lemmas 2.1 and 2.2.

Lemma 2.1 ([30]). *Consider the following fractional order system:*

$$\left.\begin{aligned}
{}_{t_0}^{c}D^q x(t) &= f(t, x),\\
x(0) &= x_0
\end{aligned}\right\} \tag{19}$$

where $f(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. The equilibrium points (14) are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $\frac{\partial f(t,x)}{\partial x}$ evaluated at the equilibrium points satisfy the following condition:

$$|arg(\lambda_i) > \frac{q\pi}{2}.$$

Lemma 2.2 ([31], *Routh–Hurwitz Criteria*). *Given the polynomial:*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + a_4\lambda^{n-4} + \cdots + a_{n-1}\lambda + a_n,$$

where the coefficients a_i are real constants, $i = 1, \ldots, n$, define the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial

$$H_1 = \begin{bmatrix} a_1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \quad H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix},$$

and

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

where $a_j = 0$ *if* $j > n$. *All of the roots of the polynomial* $P(\lambda)$ *are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive:*

$$det(H_j) > 0, \qquad j = 1, 2, \ldots, n.$$

Routh–Hurwitz criteria for $n = 2, 3,$ *and* 4 *are as follows:*

$$\begin{aligned} &(C1)\ n = 2: \quad && a_1 > 0, \quad \text{and} \quad a_2 > 0, \\ &(C2)\ n = 3: \quad && a_1 > 0, \quad a_3 > 0, \quad \text{and} \quad a_1a_2 > a_3 \\ &(C3)\ n = 4: \quad && a_1 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad \text{and} \quad a_1a_2a_3 > a_3^2 + a_1^2a_4. \end{aligned}$$

Theorem 2.2.

(i) *The trivial equilibrium point* $\mathcal{E}_0$ *is locally asymptotically unstable.*
(ii) *If* $b^q < (\mu_L^q + \alpha_L^q + ud^q)$, *then the pest-extinction equilibrium point* $\mathcal{E}_1$ *is locally asymptotically stable.*
(iii) *If* $r^q\rho < \beta^q\eta^q$ *and condition* $(C1)$ *of* Lemma 2.2 *holds, then the equilibrium point* $\mathcal{E}_2$ *is locally asymptotically stable, otherwise it is unstable.*
(iv) *If* $b^q + e\beta^q(1-u)K_M^q < (\mu_L^q + \alpha^q + ud^q)$ *and condition* $(C1)$ *of* Lemma 2.2 *holds, then the equilibrium point* $\mathcal{E}_4$ *is locally asymptotically stable, otherwise it is unstable.*
(v) *If condition* $(C2)$ *of* Lemma 2.2 *holds, then the equilibrium point* $\mathcal{E}_5$ *is locally asymptotically stable, otherwise it is unstable.*

Proof.

(i) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_0$ is

$$J(\mathcal{E}_0) = \begin{bmatrix} r^q & 0 & 0 & 0 & \\ 0 & b^q - (\mu_L^q + \alpha_L^q + ud^q) & 0 & 0 & \\ 0 & \alpha_L^q & -(\mu^q + ud^q) & 0 & \\ 0 & 0 & 0 & 0 & -\eta^q \end{bmatrix}.$$

The eigenvalues of matrix $J(\mathcal{E}_0)$ are $\lambda_1 = r^q > 0$, $\lambda_2 = b^q - (\mu_L^q + \alpha_L^q + ud^q)$, $\lambda_3 = -(\mu_A + ud^q)$ and $\lambda_4 = -\eta^q$. Since $\lambda_1 > 0$ it follows that the trivial equilibrium point $\mathcal{E}_0$ is locally asymptotically unstable.

(ii) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_1$ is

$$J(\mathcal{E}_1) \begin{bmatrix} -r^q & 0 & 0 & 0 & \\ 0 & b^q - (\mu_L^q + \alpha_L^q + ud^q) & 0 & 0 & \\ 0 & \alpha_L^q & -(\mu^q + ud^q) & 0 & \\ 0 & 0 & 0 & 0 & -\eta^q \end{bmatrix}.$$

The eigenvalues of matrix $J(\mathcal{E}_0)$ are $\lambda_1 = -r^q$, $\lambda_2 = b^q - (\mu_L^q + \alpha_L^q + ud^q)$, $\lambda_3 = -(\mu_A + ud^q)$ and $\lambda_4 = -\eta^q$. Following Lemma 2.1, it can be observed that the equilibrium point $\mathcal{E}_1$ is locally asymptotically stable if $b^q < (\mu_L^q + \alpha_L^q + ud^q)$

(iii) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_2$ is:

$$J(\mathcal{E}_2) = \begin{bmatrix} r^q - \beta^q L_2 & 0 & 0 & 0 \\ e\beta^q(1-u)L_2 & \widetilde{m} & 0 & -\sigma^q L_2 \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & \rho Z_2 & 0 & -\eta^q + \rho L_2 \end{bmatrix}. \tag{20}$$

with $\widetilde{m} = b^q - \sigma^q Z_2 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q L_2}{K_L^q}$. The eigenvalues of matrix (20) are $\lambda_1 = r^q - \frac{\beta^q\eta^q}{\rho}$, $\lambda_2 = -(\mu_A^q + ud^q)$ and the remaining eigenvalues can be obtained from the reduced matrix

$$\widetilde{J}(\mathcal{E}_2) = \begin{bmatrix} b^q - \sigma^q Z_2 - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_2}{K_L^q} & -\sigma^q L_2 \\ \rho Z_2 & -\eta^q + \rho L_2 \end{bmatrix}, \tag{21}$$

whose characteristic equation is as follows

$$\lambda^2 + a_1\lambda + a_2 = 0, \tag{22}$$

with

$$\begin{aligned} a_1 &= \eta^q + \sigma^q Z_2 + (\mu_L^q + \alpha_L^q + ud^q) - b^q, \\ &= \eta^q + \sigma^q \frac{b^q\eta^q + \rho K_L^q(\mu_L^q + \alpha^q + ud^q)}{\rho\sigma^q K_L^q}\left(\frac{b^q\rho K_L^q}{b^q\eta^q + \rho K_L^q(\mu_L^q + ud^q + \alpha^q)} - 1\right) \\ &\quad + b^q\left(\frac{(\mu_L^q + \alpha_L^q + ud^q)}{b^q} - 1\right) \\ a_2 &= \sigma^q\eta^q Z_2 + (\eta^q - \rho L_2)\left((\mu_L^q + \alpha_L^q + ud^q) + \frac{2bL_2}{K_L} - b^q\right) \\ &= \sigma^q\eta^q \frac{b^q\eta^q + \rho K_L^q(\mu_L^q + \alpha^q + ud^q)}{\rho\sigma^q K_L^q}\left(\frac{b^q\rho K_L^q}{b^q\eta^q + \rho K_L^q(\mu_L^q + ud^q + \alpha^q)} - 1\right). \end{aligned} \tag{23}$$

Therefore, if $r^q\rho < \beta^q\eta^q$ and condition $(C1)$ of Lemma 2.2 holds, then the equilibrium point $\mathcal{E}_2$ is locally asymptotically stable, otherwise it is unstable.

(iv) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_3$ is

$$J(\mathcal{E}_3) = \begin{bmatrix} r^q - \beta^q L_3 & 0 & 0 & 0 \\ e\beta^q L_3 & b^q - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_3}{K_L^q} & 0 & -\sigma^q L_3 \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & \rho Z_3 & 0 & -\eta^q + \rho L_3 \end{bmatrix}. \tag{24}$$

One can observe that, $\lambda_1 = r^q - \frac{\beta^q\eta^q}{\rho}$, $\lambda_2 = -(\mu_A^q + ud^q)$ are some of the eigenvalues of the Jacobian matrix (24), hence matrix (24) reduces to

$$\widetilde{J}(\mathcal{E}_3) = \begin{bmatrix} b^q - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_3}{K_L^q} & -\sigma^q L_3 \\ \rho Z_3 & -\eta^q + \rho L_3 \end{bmatrix}. \tag{25}$$

From (25) we have the characteristic equation

$$\lambda^2 + \bar{a}_1\lambda + \bar{a}_2 = 0, \tag{26}$$

with

$$\begin{aligned} \bar{a}_1 &= b^q\left(\frac{(\eta^q + \mu_L^q + \alpha_L^q + ud^q)}{b^q} - 1\right) + 2\eta^q\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right), \\ \bar{a}_2 &= \eta^q(\mu_L^q + \alpha^q + ud^q)\left(1 - \frac{b^q}{(\mu_L^q + \alpha^q + ud^q)}\right)\left(1 - \frac{\rho K_L^q}{b^q}\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right)\right) \\ &\quad + 2(\eta^q)^2\left(1 - \rho\frac{\eta^q K_L^q}{b^q}\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right)\right)\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right). \end{aligned} \tag{27}$$

Thus, if $r^q\rho < \beta^q\eta^q$ and condition $(C1)$ of Lemma 2.2 holds, then the equilibrium point $\mathcal{E}_3$ is locally asymptotically stable, otherwise it is unstable.

(iv) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_4$ is

$$J(\mathcal{E}_4) = \begin{bmatrix} r^q - \beta^q L_4 - \dfrac{2r^q M_4}{K_M^q} & -\beta^q M_4 & 0 & 0 \\ e\beta^q(1-u)L_4 & \bar{n} & 0 & -\sigma^q L_4 \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & 0 & 0 & -\eta^q + \rho L_4 \end{bmatrix}, \tag{28}$$

with $\bar{n} = b^q + e\beta^q(1-u)M_4 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q L_4}{K_L^q}$. The eigenvalues of $J(\mathcal{E}_4)$ are;

$$\begin{aligned}
\lambda_1 &= -(\mu_A^q + ud^q) \\
\lambda_2 &= -\eta^q + \rho L_4 \\
&= -\eta^q - \frac{\rho r^q K_L^q}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q}\left(1 - \frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha^q + ud^q)}\right).
\end{aligned} \tag{29}$$

Hence, matrix (28) reduces to

$$\widetilde{J}(\mathcal{E}_4) = \begin{bmatrix} w_1 & -\beta^q M_4 \\ e\beta^q(1-u)L_4 & w_2 \end{bmatrix}, \tag{30}$$

with

$$\begin{aligned}
w_1 &= r^q - \beta^q L_4 - \frac{2r^q M_4}{K_M^q} \\
&= r^q - \frac{\beta^q r^q K_L^q}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right) \\
&\quad - \frac{2r^q\left[r^q b^q + \beta^q K_L^q(\mu_L^q + \alpha_L^q + ud^q)\left(1 - \frac{b^q}{(\mu_L^q + \alpha_L^q + ud^q)}\right)\right]}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q}, \\
w_2 &= b^q + e\beta^q(1-u)M_4 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q L_4}{K_L^q} \\
&= b^q + e\beta^q(1-u)\frac{K_M^q\left[r^q b^q + \beta^q K_L^q(\mu_L^q + \alpha_L^q + ud^q)\left(1 - \frac{b^q}{(\mu_L^q + \alpha_L^q + ud^q)}\right)\right]}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q} \\
&\quad - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q r^q}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right).
\end{aligned}$$

From (30), the corresponding characteristic equation is

$$\lambda^2 + \tilde{a}_1\lambda + \tilde{a}_2 = 0, \tag{31}$$

with

$$\begin{aligned}
\tilde{a}_1 &= -(w_1 + w_2), \\
\tilde{a}_2 &= w_1 w_2 + \frac{e\beta^{2q}(1-u)^2 r^q K_L^q}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1\right).
\end{aligned}$$

Therefore, if $b^q + e\beta^q(1-u)K_M^q < (\mu_L^q + \alpha_L^q + ud^q)$ and condition $(C1)$ of Lemma 2.2 holds, then the equilibrium point $\mathcal{E}_4$ is locally asymptotically stable, otherwise it is unstable.

(vi) Since all the variables are non-zero at the coexistence equilibrium point, it follows that matrix J (18) is the Jacobian matrix of system (2) at this equilibrium point. From (18) one can observe, that $\lambda_1 = -(\mu_A^q + ud^q)$ and the remainder can be obtained from the following reduced matrix:

$$\bar{J}(\mathcal{E}_5) = \begin{bmatrix} \bar{w}_1 & -\beta^q M_5 & 0 \\ e\beta^q L_5 & \bar{w}_2 & -\sigma^q L_5 \\ 0 & \rho Z_5 & \bar{w}_3 \end{bmatrix}. \tag{32}$$

where

$$\begin{aligned}
\bar{w}_1 &= r^q - \beta^q L_5 - \frac{2r^q M_5}{K_M^q}, \\
\bar{w}_2 &= b^q + e\beta^q(1-u)M_5 - \sigma^q Z_5 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q L_5}{K_L^q}, \\
\bar{w}_3 &= -\eta^q + \rho L_5.
\end{aligned} \tag{33}$$

The corresponding characteristic equation at $\mathcal{E}_5$ becomes

$$\lambda^3 + a_1^*\lambda^2 + a_2^*\lambda + a_3^* = 0,$$

with

$$
\begin{aligned}
a_1^* &= -(\bar{w}_1 + \bar{w}_2 + \bar{w}_3), \\
a_2^* &= \bar{w}_1(\bar{w}_2 + \bar{w}_2) + \bar{w}_2\bar{w}_3 + \sigma\rho L_5 Z_5 + e(\beta^q)^2 L_5 M_5, \\
a_3^* &= -\bar{w}_1(\sigma^q \rho L_5 Z_5 + \bar{w}_2\bar{w}_3) - e(\beta^q)^2 L_5 M_5 \bar{w}_3.
\end{aligned}
$$

Since $\lambda_1 < 0$, it follows that condition $(C2)$ of Lemma 2.2 holds, then the equilibrium point $\mathcal{E}_5$ is locally asymptotically stable, otherwise it is unstable. This completes the proof. □

2.5. Global stability analysis of the equilibrium points

In this section, Lyapunov functions will be constructed in order to investigate the global stability of the equilibrium points of the model. To simplify the analysis, let $g_0(M) = r^q M(1 - M/K_M)$ and $g_1(L, A) = b_L(1 - L/K_L)A$.

Theorem 2.3. *The trivial equilibrium point $\mathcal{E}_0$ is globally asymptotically stable whenever:*

$$
eg_0(M) + g_1(L, A) \le \frac{(\mu_L^q + \alpha_L^q + ud^q)(\mu_A^q + ud^q)A}{\alpha_L^q} + \frac{\sigma^q \eta^q}{\rho} Z.
$$

Proof. Let us consider the following Lyapunov function:

$$
U_0(t) = eM(t) + L(t) + \frac{(\mu_L^q + \alpha_L^q + ud^q)}{\alpha_L^q} A(t) + \frac{1}{\rho} Z(t). \tag{34}
$$

The fractional derivative of (34) along the solutions of system (2) leads to:

$$
\begin{aligned}
{}_a^c D_t^q U_0(t) &\le {}_a^c D_t^q [eM(t)] + {}_a^c D_t^q L(t) + {}_a^c D_t^q \left[\frac{(\mu_L^q + \alpha_L^q + ud^q)}{\alpha_L^q} A(t) \right] + {}_a^c D_t^q \left[\frac{1}{\rho} Z(t) \right] \\
&= e[g_0(M) - \beta^q(1-u)LM] + g_1(L, A) + e\beta^q(1-u)LM - \sigma^q ZL - (\mu_L^q + \alpha_L^q + ud^q)L \\
&\quad + \frac{(\mu_L^q + \alpha_L^q + ud^q)}{\alpha_L^q} \left[\alpha_L^q L - (\mu_A^q + ud^q)A \right] + \frac{1}{\rho} \left[\rho\sigma^q LZ - \eta^q Z \right] \\
&= eg_0(M) + g_1(L, A) - \frac{(\mu_L^q + \alpha_L^q + ud^q)(\mu_A^q + ud^q)A}{\alpha_L^q} - \frac{\eta^q}{\rho} Z.
\end{aligned} \tag{35}
$$

It follows that if $M(t) = M_0$, $L(t) = L_0$, $A(t) = A_0$ and $Z(t) = Z_0$, then ${}_a^c D_t^q U_1(t) = 0$. However, if:

$$
eg_0(M) + g_1(L, A) \le \frac{(\mu_L^q + \alpha_L^q + ud^q)(\mu_A^q + ud^q)A}{\alpha_L^q} + \frac{\sigma^q \eta^q}{\rho} Z < 0,
$$

then ${}_a^c D_t^q U_1(t) < 0$ and the trivial equilibrium point $\mathcal{E}_0$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

Theorem 2.4. *The equilibrium point $\mathcal{E}_1$ is globally asymptotically stable whenever:*

$$
eg_0(M)\left(1 - \frac{M^*}{M} + \beta^q(1-u)\frac{LM^*}{g_0(M)}\right) + g_1(L, A) - \frac{(\mu_L^q + \alpha_L^q + ud^q)(\mu_A^q + ud^q)A}{\alpha_L^q} - \frac{\sigma^q \eta^q}{\rho} Z \le 0.
$$

Proof.

$$
U_1(t) = e\left[M(t) - M_1 - M_1 \ln\left(\frac{M(t)}{M_1} \right) \right] + L(t) + \frac{(\mu_L^q + \alpha_L^q + ud^q)}{\alpha_L^q} A(t) + \frac{1}{\rho^q} Z(t). \tag{36}
$$

The fractional derivative of (36) along the solutions of system (2) leads to:

$$
\begin{aligned}
{}_a^c D_t^q U_1(t) &\le e\left(1 - \frac{M^*}{M(t)}\right) {}_a^c D_t^q M(t) + {}_a^c D_t^q L(t) + {}_a^c D_t^q \left[\frac{(\mu_L + \alpha_L + ud)}{\alpha_L} A(t) \right] + {}_a^c D_t^q \left[\frac{1}{\rho} Z(t) \right] \\
&= e\left(1 - \frac{M_1}{M(t)}\right)(g_0(M) - \beta^q(1-u)LM) + g_1(L, A) + e\beta^q(1-u)LM - \sigma^q ZL
\end{aligned}
$$

$$
\begin{aligned}
&-(\mu_L+\alpha_L+ud)L+\frac{(\mu_L^q+\alpha_L^q+ud^q)}{\alpha_L^q}\left[\alpha_L^qL-(\mu_A^q+ud^q)A\right]+\frac{1}{\rho}\left[\rho\sigma^qLZ-\eta^qZ\right]\\
&=eg_0(M)\left(1-\frac{M_1}{M}+\beta(1-u)\frac{LM_1}{g_0(M)}\right)+g_1(L,A)-\frac{(\mu_L+\alpha_L+ud)(\mu_A^q+ud^q)A}{\alpha_L^q}\\
&-\frac{\sigma^q\eta^q}{\rho}Z.
\end{aligned}
$$

It follows that if $M(t)=M_1$, $L(t)=L_1$, $A(t)=A_1$ and $Z(t)=Z_1$, then ${}_a^cD_t^qU_1(t)=0$. However, if:

$$
eg_0(M)\left(1-\frac{M_1}{M}+\beta^q(1-u)\frac{LM_1}{g_0(M)}\right)+g_1(L,A)-\frac{(\mu_L^q+\alpha_L^q+ud^q)(\mu_A^q+ud)A}{\alpha_L^q}-\frac{\sigma^q\eta^q}{\rho}Z<0,
$$

then ${}_a^cD_t^qU_1(t)<0$ and the trivial equilibrium point $\mathcal{E}_1$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

Theorem 2.5. *The equilibrium point $\mathcal{E}_2$ is globally asymptotically stable whenever:*

$$
\begin{aligned}
&g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right)+L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right)\\
&+eg_0(M)-e\beta^q(1-u)L_2M\le 0.
\end{aligned}
$$

Proof. Consider the Lyapunov functional:

$$
\begin{aligned}
U_1(t)=\,&eM(t)+\left[L(t)-L_2-L_2\ln\left(\frac{L(t)}{L_2}\right)\right]+\frac{1}{\alpha_L^q}\left[A(t)-A_2-A_2\ln\left(\frac{A(t)}{A_2}\right)\right]\\
&+\frac{1}{\rho}\left[Z(t)-Z_2-Z_2\ln\left(\frac{Z(t)}{Z_2}\right)\right].
\end{aligned}\tag{37}
$$

The fractional derivative of (37) along the solutions of system (2) leads to:

$$
\begin{aligned}
{}_a^cD_t^qU_2(t)\le\,&e\,{}_a^cD_t^qM(t)+\left(1-\frac{L^3}{L(t)}\right){}_a^cD_t^qL(t)+\frac{1}{\alpha_L^q}\left(1-\frac{M^3}{M(t)}\right){}_a^cD_t^qA(t)\\
&+\frac{1}{\rho}\left(1-\frac{Z^3}{Z(t)}\right){}_a^cD_t^qZ(t).
\end{aligned}\tag{38}
$$

At the equilibrium point $\mathcal{E}_2$ we have the following identities:

$$
(\mu_L^q+\alpha_L^q+ud^q)L_2=g_1(L_2,A_2)-\sigma^qZ_2L_2,\qquad(\mu_A^q+ud^q)A_2=\alpha_L^qL_2,\qquad\eta^q=\sigma^q\rho L_2.
$$

Making use of these identities leads to

$$
\begin{aligned}
{}_a^cD_t^qU_2(t)\le\,&g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g_1(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right)+L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right)\\
&+eg_0(M)-e\beta^q(1-u)L_2M.
\end{aligned}\tag{39}
$$

We can note that, at the equilibrium point $\mathcal{E}_3$ one can easily verify that ${}_a^cD_t^qU_2(t)=0$ and ${}_a^cD_t^qU_2(t)<0$ if and only if:

$$
\begin{aligned}
&g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g_1(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right)+L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right)\\
&+eg_0(M)-e\beta^q(1-u)L_2M<0.
\end{aligned}
$$

Hence, if the above condition holds then $\mathcal{E}_2$ is globally asymptotically stable. This completes the proof. □

Theorem 2.6. *The equilibrium point $\mathcal{E}_3$ is globally asymptotically stable whenever:*

$$g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right)+L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right)$$
$$+eg_0(M)-e\beta^q(1-u)L_3M-\eta^qZ\left(1-\frac{\sigma^q}{\eta^q}L_3\right)\leq 0.$$

Proof. Consider the Lyapunov functional:

$$U_3(t)=eM(t)+\left[L(t)-L_3-L_3\ln\left(\frac{L(t)}{L_3}\right)\right]+\frac{1}{\alpha_L^q}\left[A(t)-A_3-A_3\ln\left(\frac{A(t)}{A_3}\right)\right]$$
$$+\frac{1}{\rho}Z(t). \tag{40}$$

At the equilibrium point $\mathcal{E}_3$ we have the identities:

$$(\mu_L^q+\alpha_L^q+ud^q)L_3=g_1(L_3,A_3),\qquad (\mu_A^q+ud^q)A_3=\alpha_L^qL_3.$$

Utilizing these identities leads to the following result:

$${}_a^cD_t^qU_3(t)\leq g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right)+L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right)$$
$$+eg_0(M)-e\beta^q(1-u)L_3M-\eta^q\left(1-\frac{\sigma^q}{\eta^q}L_3\right).$$

It follows that if $M(t)=M_3$, $L(t)=L_3$, $A(t)=A_3$ and $Z(t)=Z_3$, then ${}_a^cD_t^qU_3(t)=0$. However, if:

$$g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right)+L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right)$$
$$+eg_0(M)-e\beta^q(1-u)L_3M-\eta^q\left(1-\frac{\sigma^q}{\eta^q}L_3\right)<0,$$

then ${}_a^cD_t^qU_3(t)<0$ and it follows that equilibrium point $\mathcal{E}_3$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

Theorem 2.7. *The equilibrium point $\mathcal{E}_4$ is globally asymptotically stable whenever:*

$$g_0(M_4)\left(\frac{L}{L_4}+\frac{g_0(M)}{g_0(M_4)}-\frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)}-\frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)+L_4\left(1+\frac{L}{L_4}-\frac{A}{A_4}-\frac{LA_4}{L_4A}\right)$$
$$+g_1(L_4,A_4)\left(1+\frac{g_1(L,A)}{g_1(L_4,A_4)}-\frac{L}{L_4}-\frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)$$
$$+e\beta^q(1-u)L_4M_4\left(1+\frac{LM}{L_4M_4}-\frac{L}{L_4}-\frac{L_4g_1(L,A)}{Lg(L_4,A_4)}\right)\leq 0. \tag{41}$$

Proof. Consider the Lyapunov functional:

$$U_4(t)=+\left[M(t)-M_4-M_4\ln\left(\frac{M(t)}{M_4}\right)\right]+\left[L(t)-L_4-L_4\ln\left(\frac{L(t)}{L_4}\right)\right]$$
$$+\frac{1}{\alpha_L^q}\left[A(t)-A_4-A_4\ln\left(\frac{A(t)}{A_4}\right)\right]+\frac{1}{\rho}\left[Z(t)-Z_4-Z_4\ln\left(\frac{Z(t)}{Z_4}\right)\right]. \tag{42}$$

At the equilibrium point $\mathcal{E}_4$ we have the following identities:

$$g_0(M_4)=\beta^q(1-u)L_4M_4,$$
$$g_1(L_4,A_4)+e\beta^q(1-u)L_4M_4-\sigma^qL_4Z_4=(\mu_L^q+\alpha_L^q+ud^q)L_4,$$
$$(\mu_A^q+ud^q)A_4=\alpha_L^qL_4,\qquad \sigma^q\rho L_4=\eta.$$

Utilizing these identities leads to the following result:

$$\begin{aligned}
{}_a^cD_t^qU_4(t) \leq\ & g_0(M_4)\left(\frac{L}{L_4}+\frac{g_0(M)}{g_0(M_4)}-\frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)}-\frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)\\
&+g_1(L_4,A_4)\left(1+\frac{g_1(L,A)}{g_1(L_4,A_4)}-\frac{L}{L_4}-\frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)\\
&+e\beta^q(1-u)L_4M_4\left(1+\frac{LM}{L_4M_4}-\frac{L}{L_4}-\frac{L_4g_1(L,A)}{Lg(L_4,A_4)}\right)\\
&+L_4\left(1+\frac{L}{L_4}-\frac{A}{A_4}-\frac{LA_4}{L_4A}\right).
\end{aligned}$$

It follows that if $M(t)=M_4$, $L(t)=L_4$, $A(t)=A_4$ and $Z(t)=Z_4$, then ${}_a^cD_t^qU_4(t)=0$. However, if:

$$\begin{aligned}
& g_0(M_4)\left(\frac{L}{L_4}+\frac{g_0(M)}{g_0(M_4)}-\frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)}-\frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)\\
&+g_1(L_4,A_4)\left(1+\frac{g_1(L,A)}{g_1(L_4,A_4)}-\frac{L}{L_4}-\frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)\\
&+e\beta^q(1-u)L_4M_4\left(1+\frac{LM}{L_4M_4}-\frac{L}{L_4}-\frac{L_4g_1(L,A)}{Lg(L_4,A_4)}\right)\\
&+L_4\left(1+\frac{L}{L_4}-\frac{A}{A_4}-\frac{LA_4}{L_4A}\right)<0,
\end{aligned}$$

then ${}_a^cD_t^qU_4(t)<0$ and it follows that equilibrium point $\mathcal{E}_4$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

3. Optimal control problem

In this section, we investigate the role of time-dependent intervention strategies on minimizing the growth of the FAW population during an outbreak. Precisely, we investigate the effects of time dependent awareness campaigns as an intervention to control the growth of FAW population. Hence the constant awareness campaign parameter u in model (2) is now considered to be time-dependent, that is, $0 \leq u(t) \leq u_{\max} < 1$, where $u_{\max}$ is the upper bound of the control $u(t)$, which reflects practical limitation on the maximum rate of control that can be implemented in a given period. In what follows, we introduce an objective functional J which will be utilized to formulate the optimization problem of interest. In particular, the overall objective here is to minimize the number of FAW larvae and moths over a finite time interval $[0, T]$ at minimal costs. Mathematically, this can be captured as follows:

$$J[u(t)]=\min_{\Omega}\int_0^T\left[L(t)+A(t)+\frac{W}{2}u^2(t)\right]dt, \tag{43}$$

subject to the system:

$$\left.\begin{aligned}
{}_a^cD_t^qM(t) &= r^qM\left(1-\frac{M}{K_M^q}\right)-\beta^q(1-u(t))LM,\\
{}_a^cD_t^qL(t) &= b_L^qA\left(1-\frac{L}{K_L^q}\right)+e\beta^q(1-u(t))LM-\sigma^qZL-(\mu_L^q+\alpha_L^q+u(t)d^q)L,\\
{}_a^cD_t^qA(t) &= \alpha_L^qL-(\mu_A^q+u(t)d^q)A,\\
{}_a^cD_t^qZ(t) &= \rho\sigma^qLZ-\eta^qZ.
\end{aligned}\right\} \tag{44}$$

In Eq. (43), W is known as the weight constant. The weight constant over the prescribed time frame is a measure of the relative costs of the interventions over a finite time horizon. The optimal control problem hence becomes that, we seek an optimal function, $u^*(t)$, such that $J(u^*(t)) = \min_{\Omega} J(u(t))$ subject to the state equations in system (44) with initial conditions (2).

3.1. Optimality system

We use Pontryagin's maximum principle [32,33] to determine the necessary conditions that optimal controls must satisfy. Through Pontryagin's maximum principle, system (44) is converted into an equivalent problem, namely the problem of minimizing the Hamiltonian $H(t)$ given by:

$$\left.\begin{aligned} H(t) &= L(t)+A(t)+\frac{W}{2}u^2(t) \\ &\quad +\lambda_1\left[r^qM\left(1-\frac{M}{K_M^q}\right)-\beta^q(1-u(t))LM\right] \\ &\quad +\lambda_2\left[b_L^qA\left(1-\frac{L}{K_L^q}\right)+e\beta^q(1-u(t))LM-\sigma^qZL-(\mu_L^q+\alpha_L^q+u(t)d^q)L\right] \\ &\quad +\lambda_3\left[\alpha_L^qL-(\mu_A^q+u(t)d^q)A\right]+\lambda_4\left[\rho\sigma^qLZ-\eta^qZ\right], \end{aligned}\right\} \tag{45}$$

where $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ are $\lambda_4(t)$ are the adjoint variables corresponding to the states $M(t)$, $L(t)$, $A(t)$ and $Z(t)$.

Given an optimal control $u^*(t)$ and the corresponding state solutions M, L, A and Z, there exist adjoint functions $\lambda_i(t)$, $i = 1, 2, 3, 4$ satisfying:

$$\left.\begin{aligned} {}_a^cD_t^q\lambda_1(T-t) &= \left[r^q-\frac{2r^qM(T-t)}{K_M^q}-\beta^q(1-u(T-t))L(T-t)\right]\lambda_1(T-t) \\ &\quad +e\beta^q(1-u(T-t))L(T-t)\lambda_2(T-t), \\ {}_a^cD_t^q\lambda_2(T-t) &= 1-\beta^q(1-u(T-t))M(T-t)\lambda_1(T-t)+\alpha_L^q\lambda_3(T-t) \\ &\quad +\sigma^q\rho Z(T-t)\lambda_4(T-t)+e\beta^q(1-u(T-t))M(T-t)\lambda_2(T-t) \\ &\quad -\left[\alpha_L^q+\mu_L^q+u(T-t)d^q-\frac{b_L^qA(T-t)}{K_L}+\sigma^qZ(T-t)\right]\lambda_2(T-t), \\ {}_a^cD_t^q\lambda_3(T-t) &= 1-(\mu_A^q+u(T-t)d^q)\lambda_3(T-t)+b_L\left(1-\frac{L(T-t)}{K_L^q}\right)\lambda_2(T-t), \\ {}_a^cD_t^q\lambda_4(T-t) &= -\sigma^qL(T-t)\lambda_2(T-t)+(\sigma^q\rho L(T-t)-\eta^q)\lambda_4(T-T), \end{aligned}\right\} \tag{46}$$

with transversality conditions $\lambda_i(T) = 0$ for $i = 1, 2, 3, 4$. Furthermore, the optimal controls are characterized by the optimality conditions:

$$u(t) = \min\left\{\max\left\{0, \frac{(e\beta^qM+d^q)L\lambda_2+d^qA\lambda_3-\beta^qLM\lambda_1}{W}\right\}, u_{\max}\right\}. \tag{47}$$

4. Numerical results and discussions

In this section, we present some numerical results to support the analytical results presented in Sections 2 (2.2, 2.3, 2.4. 2.5) and 3. For the numerical simulations, we use a forward–backward sweep iterative scheme [33]. The initial population levels were assumed as follows: $M(0) = 15$, $L(0) = 500$, $A(0) = 100$, and $Z(0) = 50$. All simulation of the model (2) was done using the baseline values for model parameters presented in Table 1 obtained from different literature.

Before investigating the effects of time-dependent farming awareness on minimizing or eradicating FAW in the maize field, we first simulate the model system (2) with constant awareness campaigns u. From the simulation in Fig. 2, we can observe that at this level of farming awareness ($u = 0.1$), the maize biomass will increase from the start and converge to 35 biomass per plant which is less than the expected 50 biomass per plant. This suggests that while farming awareness may minimize the effects of FAW on maize biomass, to some extent it cannot be highly effective towards achieving the expected biomass per plant. However, in Fig. 3 we can observe that if $u = 0.7$, then the level of maize biomass converges to the expected level even at different fractional order values. Thus, as the awareness level increases to levels close to 100% ($u = 1$), the FAW population decreases significantly and the final maize biomass reaches expected levels.

Next, we investigate the effects of time-dependent awareness campaigns $u(t)$ on minimizing the damage on maize biomass by FAW. Without loss of generality, we set $q = 0.9$ and $u(t) = 0.03$ per day with an upper bound of $u_{\max} = 1$. The simulation results are presented in Fig. 4.

From the results in Fig. 4, one can note that in the presence of time-dependent farming awareness, the FAW population (larvae and moth) decreases remarkably compared to when there is no time dependent farming awareness. We also note

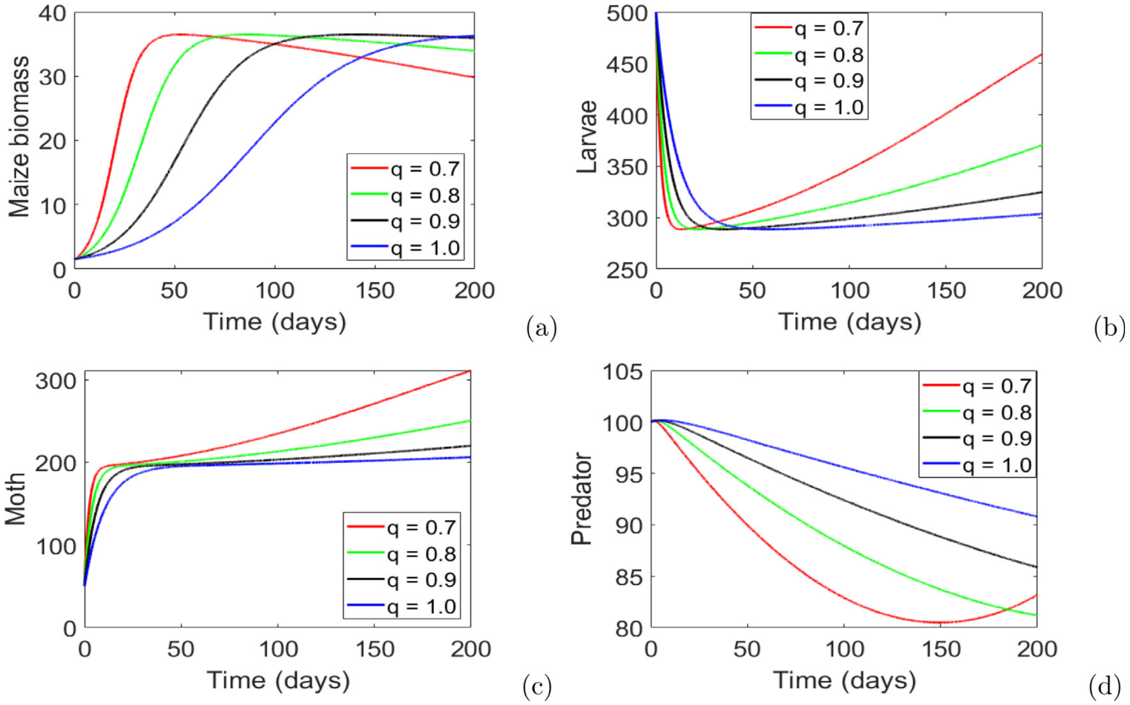

Fig. 2. Simulation results of model (2) with constant farming awareness $u = 0.1$ and different fractional order values.

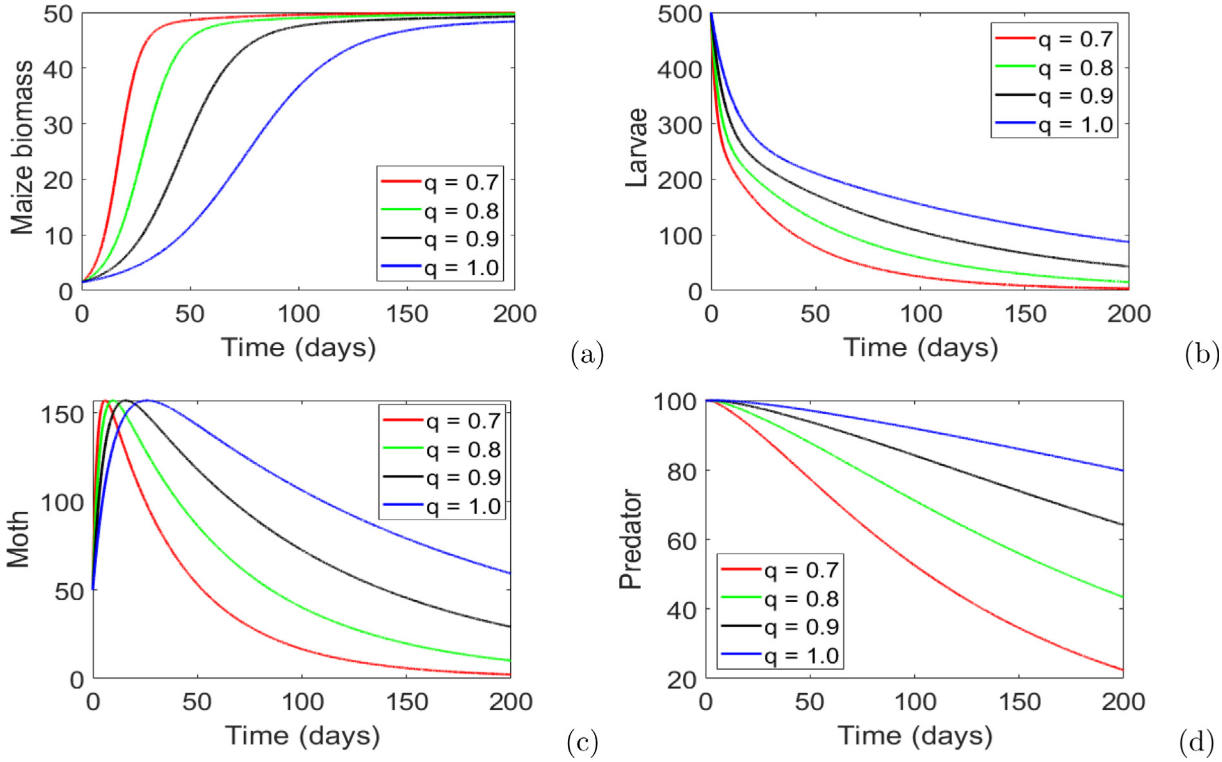

Fig. 3. Simulation results of model (2) with constant farming awareness $u = 0.7$ and different fractional order values.

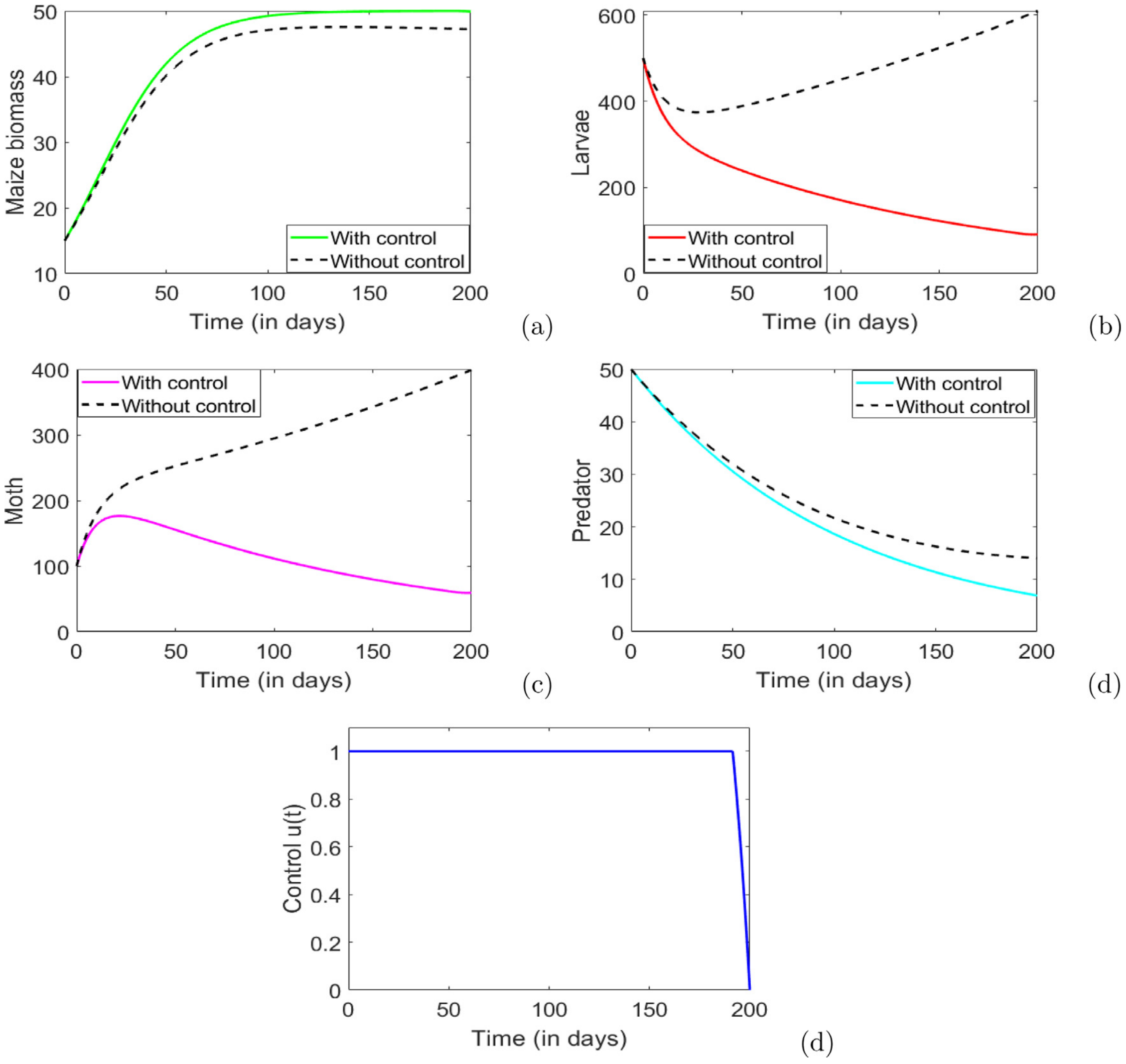

Fig. 4. Simulation results of model (2) with time-dependent constant farming awareness $0 \le u(t) \le 1$, $q = 0.9$ and $W = 10$.

Table 1
Model parameters and their baseline values.

Symbol	Definition	Baseline value	Source
b_L	Growth rate of larva	1/14 day^{-1}	[34]
α_L^{-1}	Average development time of the larva	30 Days	[34]
μ_A^{-1}	Average moth life span	21 Days	[34]
K_M	Maximum biomass of maize plants	50 plant^{-1}	[35].
K_L	Egg environmental carrying capacity	10^6	[35].
μ_L	Natural mortality rate of larva	0.01 Day^{-1}	[35].
r	Growth rate of maize plants	0.05 Day^{-1}	[35].
e	Efficiency of biomass conversion	0.2	[35].
β	Plant attack rate by larvae	5×10^{-5} Day^{-1}	[36].
σ	Consumption rate of larva by predators	5×10^{-5} Day^{-1}	[37].
ρ	Conversion rate of prey to predator	0.1 Day^{-1}	[38].
d	Mortality of FAW due to intervention strategies	0.01 Day^{-1}	[37].
η^{-1}	Average life span of predator	100 Days	[39].

that a significant decrease of the FAW larvae in the presence of optimal farming awareness will also lead to a slight decrease of the predator population over time. The results also show that in the presence of optimal farming awareness, the final maize biomass will be within the expected level. However, in the absence of optimal farming awareness the final

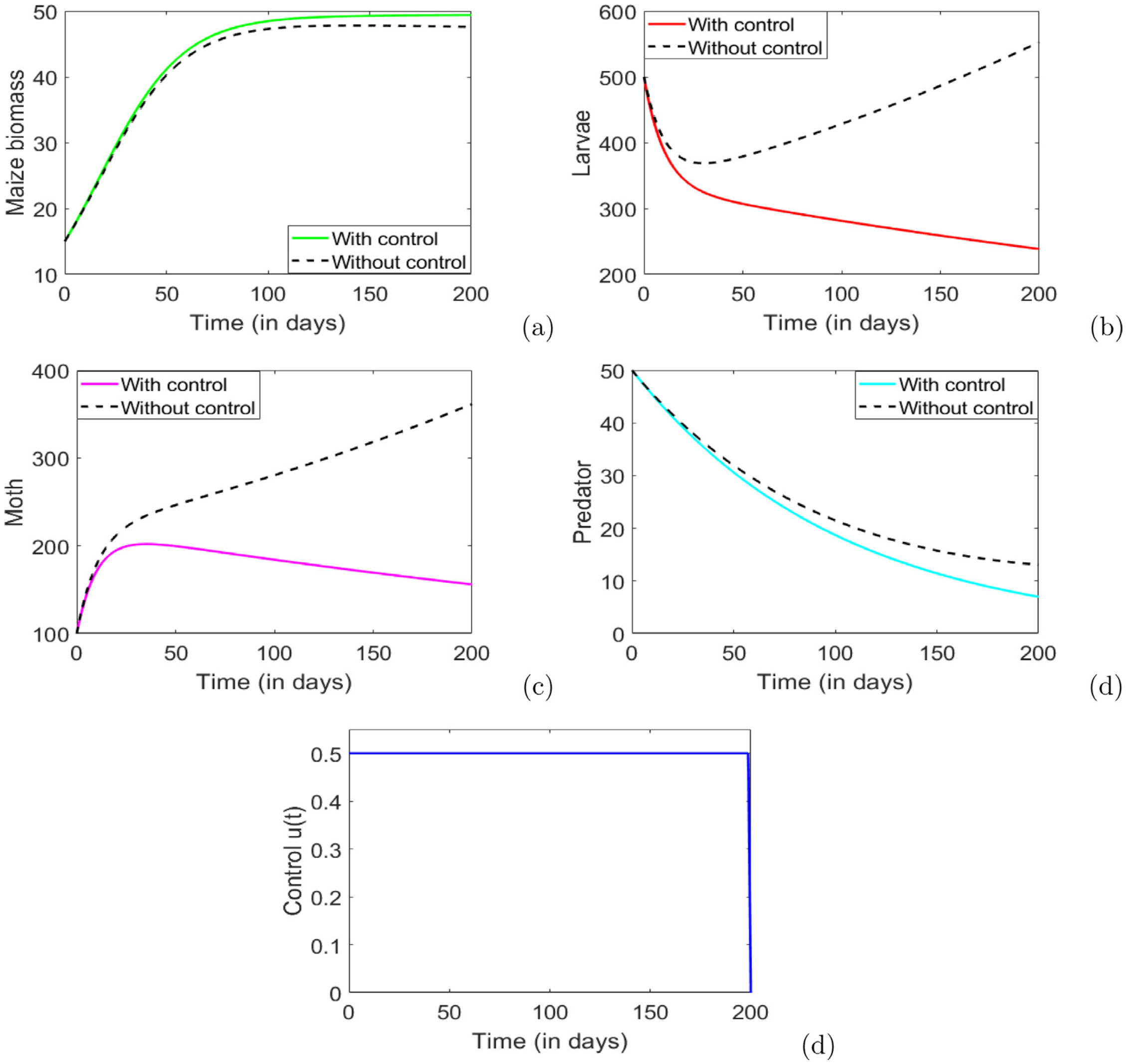

Fig. 5. Simulation results of model (2) at low maximum intensity $u_{\max} = 0.5$, with $q = 0.9$ and $W = 100$.

biomass level will always be less than the expected final biomass. In addition, one can observe that the optimal control profile (Fig. 4(d)) starts at $u_{\max} = 1$ and remains there for the greater part of time horizon ($0 \leq t \leq 195$ days) till it drops close to the final period. This suggests that for one to attain the outcomes in Fig. 4, optimal farming awareness efforts need to be maintained at their maximum intensity for the greater part of the time horizon and thereafter ceased gradually till the final time.

The simulation results in Fig. 5 show the impact of the upper bound of the control variable $u_{\max}$ on model solutions. Here, we set $u_{\max} = 0.5$. We can note that in this scenario, the optimal efforts will need to be maintained at their maximum intensity throughout the entire time horizon in order for the final maize biomass to be within the expected level.

The simulation results in Fig. 6 show that the impact of the costs on the implementation of optimal farming awareness. Here we set $W = 1000$. We note that when the costs of implementing farming awareness are high, the control profile for $u(t)$ does not start at its maximum, $u_{\max} = 1$, but begins on $u(t) = 0.8$, followed by a gradual decrease before it stabilizes at $u(t) = 0.4$ after approximately 40 days from the start. The control profile stays at $u(t) = 0.4$ till the 150th day after which it increases slightly to $u(t) = 0.5$ and immediately drops gradually to its minimum until the final time horizon. Although the pattern of the control profile is complex, one can deduce that optimum results can be attained if the intensity of the control $u(t)$ is maintained between 0.4 and 0.5 ($0.4 \leq u(t) \leq 0.5$) for a greater part of the time horizon.

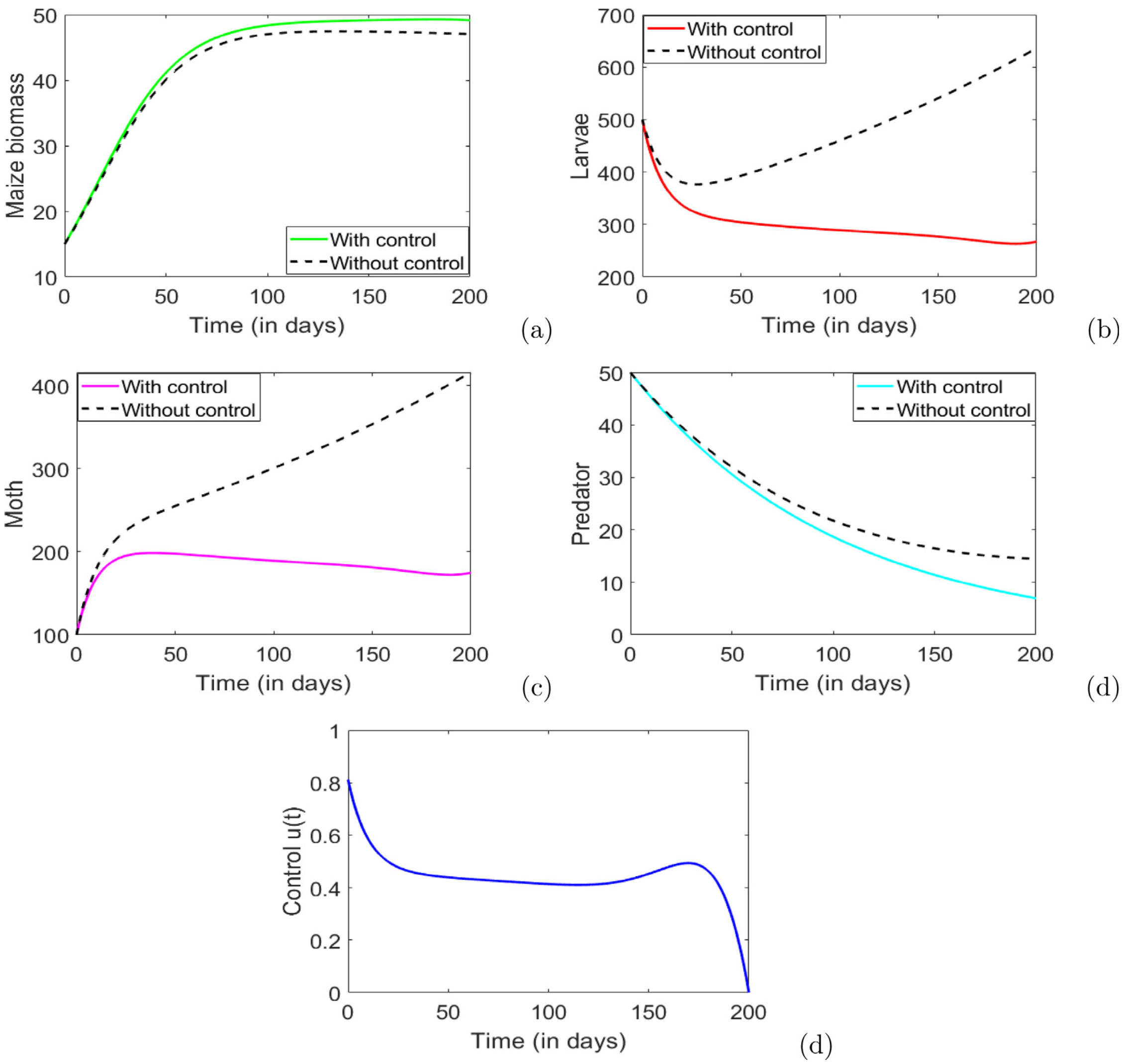

Fig. 6. Simulation results of model (2) at high cost of implementation, $W = 1000$, $0 \leq u(t) \leq 1$, and $q = 0.9$.

5. Concluding remarks

We have formulated a fractional-order model that incorporates naturally beneficial insects and optimal farming awareness. Dynamical analysis of the proposed model revealed that it has six equilibrium points which are all locally and globally asymptotically stable if the conditions outlined in Lemmas 2.1 and 2.2 are met. The simulation results for the model with constant awareness campaigns u, showed that $u = 0.7$ may lead to the achievement of the expected maize biomass at the end of the season (that is $t = 160$ days) for fractional-order values $q = 0.7, 0.8, 0.9$. However for $q = 1.0$, the final maize biomass at this level of awareness will be slightly less than the expected. For time-dependent farming awareness, we observed that the expected maize biomass can be attained if the costs of implementing the strategy are low. In addition, we observed that if the intensity of implementing is low, then the efforts can be carried out at their maximum intensity throughout the time horizon but when costs are high, the control profile for $u(t)$ does not start at its maximum, $u_{\max} = 1$, rather at $u(t) = 0.8$ followed by a gradual decrease before it stabilizes at $u(t) = 0.4$ after approximately 40 days from the start. Although this study is not exhaustive, it has illustrated the value of optimal control theory as tool to suggest effective management strategies during FAW outbreaks. In future, we will explore the effects of temperature and seasonal variation, migration of the moth and include the parameter of continuous replanting of maize crops on the dynamics of FAW and its implications on maize biomass.

CRediT authorship contribution statement

Salamida Daudi: Model formulation, Analysis. **Livingstone Luboobi:** Supervision, Writing – review. **Moatlhodi Kgosimore:** Supervision, Writing – review. **Dmitry Kuznetsov:** Supervision, Writing – review.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

The data used to support the findings of this study are included within the article.

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